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FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 21 May 2014 (afternoon)

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[150 marks]*.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 6]*

Find the positive square root of the base 7 number $(551662)_7$, giving your answer as a base 7 number.

2. *[Maximum mark: 7]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - x^3$ for which $y = 1$ when $x = 0$. Use Euler's method with a step length of 0.1 to find an approximation for the value of y when $x = 0.4$.

3. *[Maximum mark: 6]*

The following table shows the probability distribution of the discrete random variable X.

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) Show that the probability generating function of X is given by

$$G(t) = \frac{t(1+t)^2}{4}.$$ *[2]*

(b) Given that $Y = X_1 + X_2 + X_3 + X_4$, where X_1, X_2, X_3, X_4 is a random sample from the distribution of X,

(i) state the probability generating function of Y;

(ii) hence find the value of $\mathrm{P}(Y = 8)$. *[4]*

4. *[Maximum mark: 12]*

The matrix $\boldsymbol{M}$ is defined by $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The eigenvalues of $\boldsymbol{M}$ are denoted by λ_1, λ_2.

(a) Show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \lambda_2 = \det(\boldsymbol{M})$. *[3]*

(b) Given that $a + b = c + d = 1$, show that 1 is an eigenvalue of $\boldsymbol{M}$. *[2]*

(c) Find eigenvectors for the matrix $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$. *[7]*

5. *[Maximum mark: 7]*

(a) Assuming the Maclaurin series for e^x, determine the first three non-zero terms in the Maclaurin expansion of $\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$. *[3]*

(b) The random variable X has a Poisson distribution with mean μ. Show that $\mathrm{P}(X \equiv 1 \pmod 2) = a + b\mathrm{e}^{c\mu}$ where a, b and c are constants whose values are to be found. *[4]*

6. *[Maximum mark: 9]*

The parabola P has equation $y^2 = 4ax$. The distinct points $\mathrm{U}(au^2, 2au)$ and $\mathrm{V}(av^2, 2av)$ lie on P, where $u, v \neq 0$. Given that $\mathrm{U\hat{O}V}$ is a right angle, where O denotes the origin,

(a) show that $v = -\frac{4}{u}$; *[3]*

(b) find expressions for the coordinates of W, the midpoint of [UV], in terms of a and u; *[2]*

(c) show that the locus of W, as u varies, is the parabola P' with equation $y^2 = 2ax - 8a^2$; *[2]*

(d) determine the coordinates of the vertex of P'. *[2]*

7. *[Maximum mark: 11]*

The weights, in grams, of 10 apples were measured with the following results:

212.2 216.9 209.0 215.5 215.9 213.5 208.9 213.8 216.4 209.9

You may assume that this is a random sample from a normal distribution with mean μ and variance σ^2.

(a) Giving all your answers correct to four significant figures,

(i) determine unbiased estimates for μ and σ^2;

(ii) find a 95% confidence interval for μ. *[5]*

Another confidence interval for μ, [211.5, 214.9], was calculated using the above data.

(b) Find the confidence level of this interval. *[6]*

8. *[Maximum mark: 12]*

The group $\{G, *\}$ has a subgroup $\{H, *\}$. The relation R is defined, for $x, y \in G$, by xRy if and only if $x^{-1} * y \in H$.

(a) Show that R is an equivalence relation. *[8]*

(b) Given that $G = \{0, \pm 1, \pm 2, \ldots\}$, $H = \{0, \pm 4, \pm 8, \ldots\}$ and * denotes addition, find the equivalence class containing the number 3. *[4]*

9. *[Maximum mark: 5]*

ABCDEF is a hexagon. A circle lies inside the hexagon and touches each of the six sides. Show that $\text{AB} + \text{CD} + \text{EF} = \text{BC} + \text{DE} + \text{FA}$.

10. *[Maximum mark: 12]*

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

(a) Given that $\boldsymbol{A}^3$ can be expressed in the form $\boldsymbol{A}^3 = a\boldsymbol{A}^2 + b\boldsymbol{A} + c\boldsymbol{I}$, determine the values of the constants a, b, c. *[7]*

(b) (i) Hence express $\boldsymbol{A}^{-1}$ in the form $\boldsymbol{A}^{-1} = d\boldsymbol{A}^2 + e\boldsymbol{A} + f\boldsymbol{I}$ where $d, e, f \in \mathbb{Q}$.

(ii) Use this result to determine $\boldsymbol{A}^{-1}$. *[5]*

11. *[Maximum mark: 9]*

The random variables X, Y follow a bivariate normal distribution with product moment correlation coefficient ρ. The following table gives a random sample from this distribution.

x	5.1	3.8	3.7	2.5	4.0	3.7	1.6	2.8	3.3	2.9
y	4.6	4.9	4.1	5.9	4.2	1.6	5.1	2.1	6.4	4.7

(a) Determine the value of r, the product moment correlation coefficient of this sample. *[2]*

(b) (i) Write down hypotheses in terms of ρ which would enable you to test whether or not X and Y are independent.

(ii) Determine the p-value of the above sample and state your conclusion at the 5% significance level. Justify your answer. *[5]*

(c) (i) Determine the equation of the regression line of y on x.

(ii) State whether or not this equation can be used to obtain an accurate prediction of the value of y for a given value of x. Give a reason for your answer. *[2]*

12. *[Maximum mark: 11]*

Consider the infinite series $S = \sum_{n=1}^{\infty} \frac{x^n}{2^{2n}\left(2n^2-1\right)}$.

(a) Determine the radius of convergence. *[4]*

(b) Determine the interval of convergence. *[7]*

13. *[Maximum mark: 9]*

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x, y) = \left(xy, \frac{x}{y}\right)$.

Prove that f is a bijection.

14. *[Maximum mark: 12]*

(a) The function g is defined by $g(x, y) = x^2 + y^2 + dx + ey + f$ and the circle C_1 has equation $g(x, y) = 0$.

(i) Show that the centre of C_1 has coordinates $\left(-\frac{d}{2}, -\frac{e}{2}\right)$ and the radius of C_1 is $\sqrt{\frac{d^2}{4} + \frac{e^2}{4} - f}$.

(ii) The point $\mathrm{P}(a, b)$ lies outside C_1. Show that the length of the tangents from P to C_1 is equal to $\sqrt{g(a, b)}$. *[6]*

(b) The circle C_2 has equation $x^2 + y^2 - 6x - 2y + 6 = 0$.

The line $y = mx$ meets C_2 at the points R and S.

(i) Determine the quadratic equation whose roots are the x-coordinates of R and S.

(ii) **Hence**, given that L denotes the length of the tangents from the origin O to C_2, show that $\mathrm{OR} \times \mathrm{OS} = L^2$. *[6]*

15. *[Maximum mark: 12]*

(a) Show that the solution to the linear congruence $ax \equiv b \pmod{p}$, where $a, x, b, p \in \mathbb{Z}^+$, p is prime and a, p are relatively prime, is given by $x \equiv a^{p-2} b \pmod{p}$. *[4]*

(b) Consider the congruences

$$7x \equiv 13 \pmod{19}$$
$$2x \equiv 1 \pmod{7}.$$

(i) Use the result in (a) to solve the first congruence, giving your answer in the form $x \equiv k \pmod{19}$ where $1 \le k \le 18$.

(ii) Find the set of integers which satisfy both congruences simultaneously. *[8]*

16. *[Maximum mark: 10]*

$\{G, *\}$ is a group of order N and $\{H, *\}$ is a proper subgroup of $\{G, *\}$ of order n.

(a) Define the right coset of $\{H, *\}$ containing the element $a \in G$. *[1]*

(b) Show that each right coset of $\{H, *\}$ contains n elements. *[2]*

(c) Show that the union of the right cosets of $\{H, *\}$ is equal to G. *[2]*

(d) Show that any two right cosets of $\{H, *\}$ are either equal or disjoint. *[4]*

(e) Give a reason why the above results can be used to prove that N is a multiple of n. *[1]*